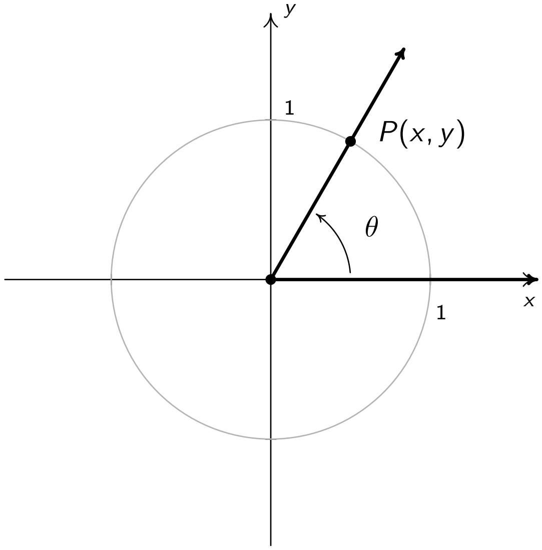
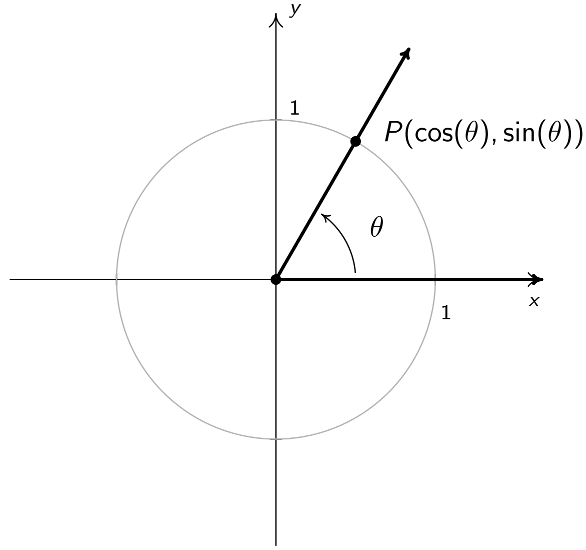
MATH 1700: SECTION 10.2: SINE AND COSINE

DEFINING SINE AND COSINE OF GENERAL ANGLES:

Consider an angle θ in standard position and let P denote the point where the terminal side of θ intersects the Unit Circle. By associating the point P with the angle θ, we are assigning a *position* on the Unit Circle to the angle θ. Since for each angle θ, the terminal side of θ, when graphed in standard position, intersects The Unit Circle only once, the mapping of θ to P is a function. Since there is only *one* way to describe a point using rectangular coordinates, the mappings of θ to each of the x and y coordinates of P are also functions.



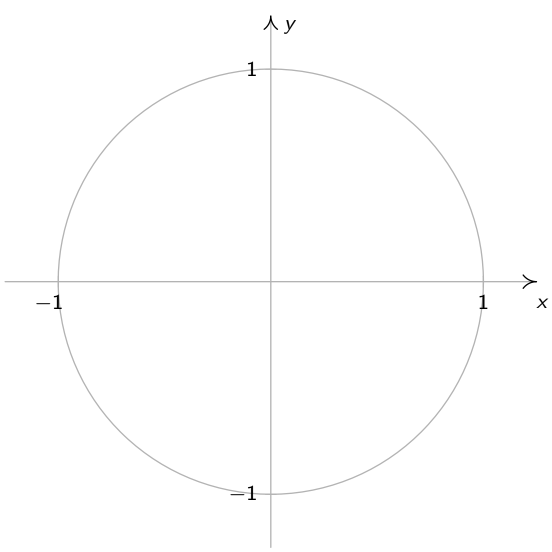


DEFINITION: Suppose an angle θ is graphed in standard position. Let $P(x, y)$ be the point of intersection of the terminal side of P and the Unit Circle.

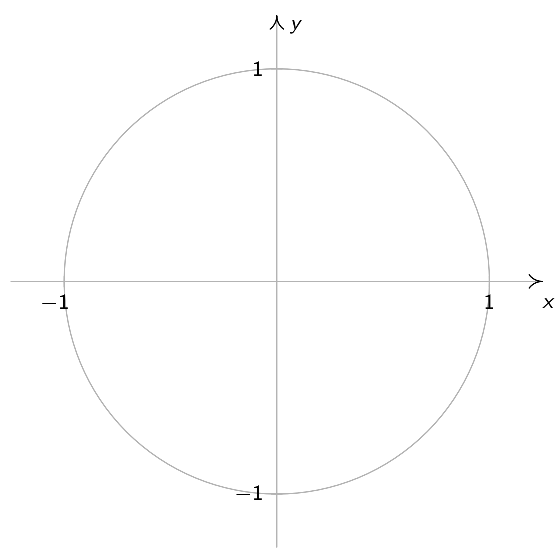
- The x-coordinate of P is called the **cosine** of θ, written $\cos(\theta)$.
- The y-coordinate of P is called the **sine** of θ, written $\sin(\theta)$.

EXAMPLE 1: Find the sine and cosine of the following angles.

- $\theta = 90^\circ$

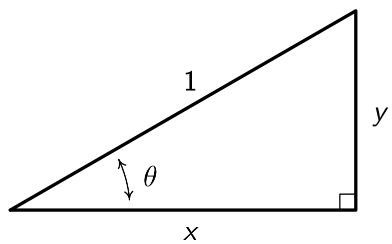


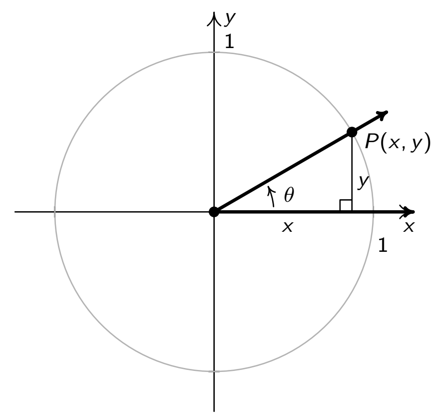
- $\theta = -\pi$



REFERENCE ANGLES:

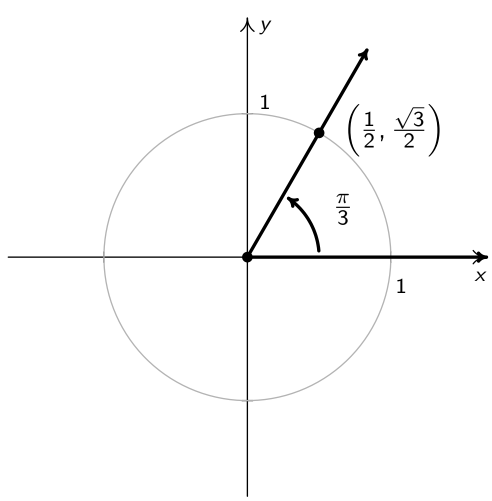
We can re-use much of our work from Right Triangle Trigonometry to help us find the sines and cosines of angles using this 'new' definition. If θ is an acute angle, we situate θ in a right triangle with hypotenuse length 1, adjacent side length 'x,' and the opposite side length 'y' as seen below on the left. Placing the vertex of θ at the origin and the adjacent side of θ along the x-axis as seen below on the right effectively puts θ in standard position with θ's adjacent side as the initial side of θ and the hypotenuse as the terminal side of θ. Since the hypotenuse of the triangle has length 1, we know the point $P(x, y)$ is on the Unit Circle.



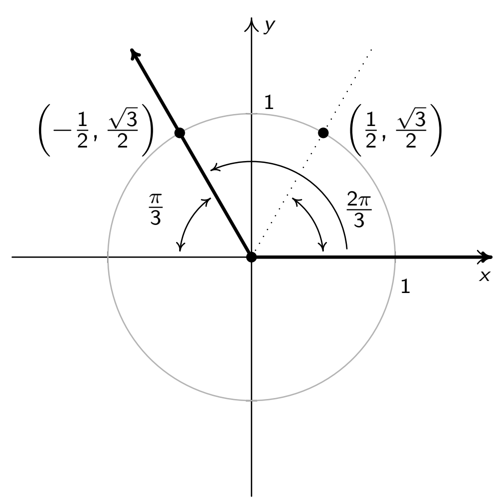


Using our definitions from right triangle trigonometry, $\cos(\theta) = \frac{x}{1} = x$ and $\sin(\theta) = \frac{y}{1} = y$ which matches our more general definition presented here. Hence, in the case of acute angles, the two definitions agree. Hence, values of the *trigonometric ratios* of acute angles are the same as the corresponding *circular function* values.

Knowing the cosine and sine values for acute angles can help us find the cosine and sine values for many other angles around the circle using symmetry. For instance, suppose we wanted to know the cosine and sine of $\theta = \dfrac{2\pi}{3} = 120^\circ$. We can use what we know about the cosine and sine of $60^\circ = \dfrac{\pi}{3}$ to help us:



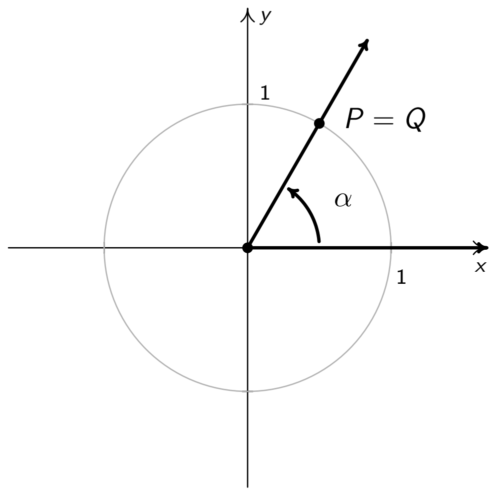
Cosine and sine of $\dfrac{\pi}{3}$



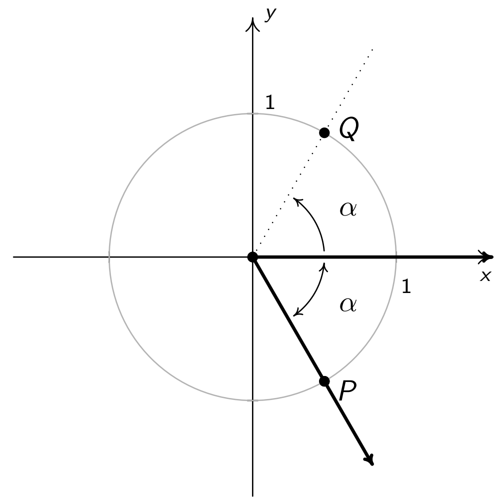
Determining cosine and sine for $\dfrac{2\pi}{3}$

We get $\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

In general, for a non-quadrantal angle θ, the reference angle for θ (which we'll usually denote α) is the *acute* angle made between the terminal side of θ and the x-axis. If θ is a Quadrant I or IV angle, α is the angle between the terminal side of θ and the *positive* x-axis:

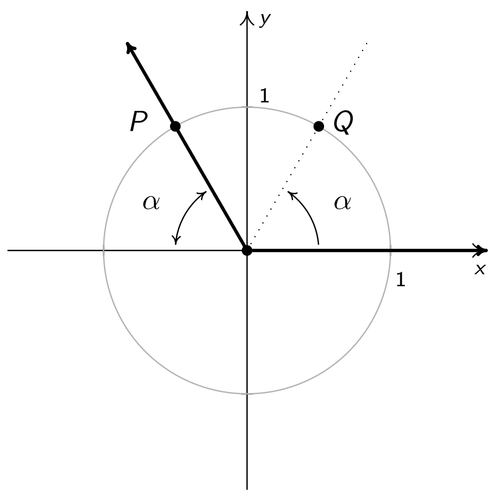


Reference angle α for a Quadrant I angle

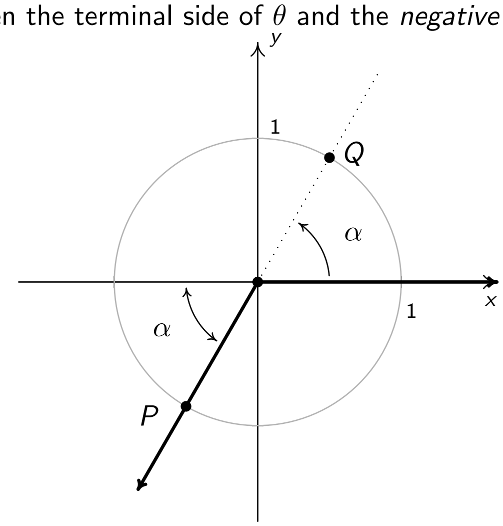


Reference angle α for a Quadrant IV angle

If θ is a Quadrant II or III angle, α is the angle between the terminal side of θ and the *negative* x-axis:



Reference angle α for a Quadrant II angle



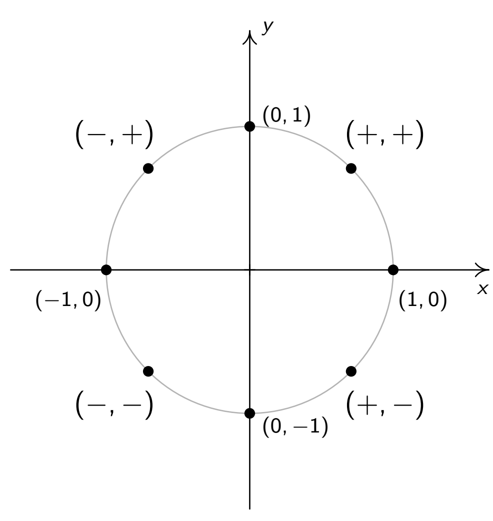
Reference angle α for a Quadrant III angle

REFERENCE ANGLE THEOREM: Suppose α is the reference angle for θ. Then:

$$\cos(\theta) = \pm\cos(\alpha) \text{ and } \sin(\theta) = \pm\sin(\alpha),$$

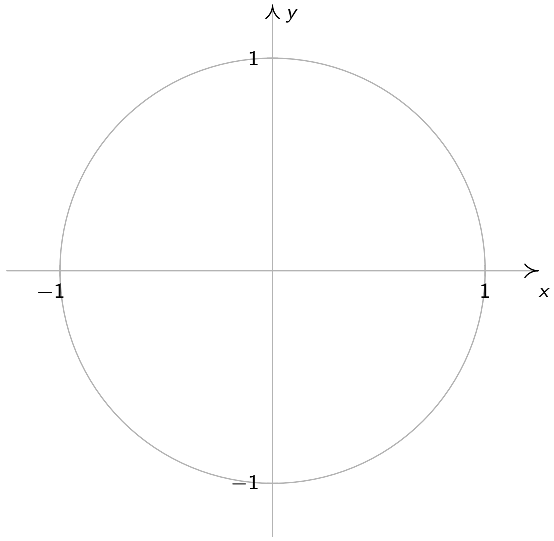
where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

θ(degrees)	θ(radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1

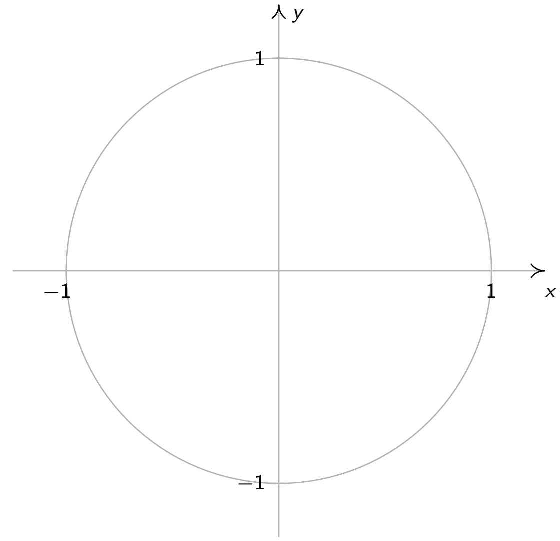


EXAMPLE 2: Find the sine and cosine of the following angles.

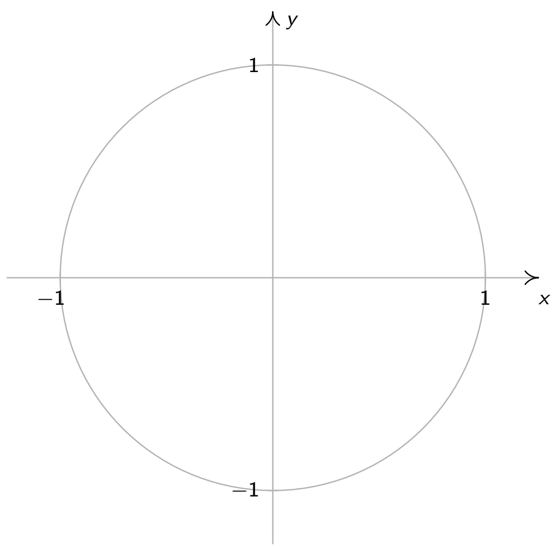
- $\theta = 225°$



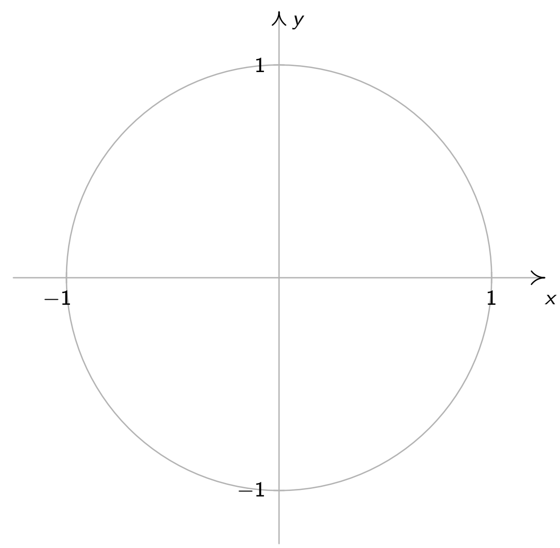
- $\theta = \frac{11\pi}{6}$



- $\theta = -\frac{5\pi}{4}$



- $\theta = \frac{7\pi}{3}$



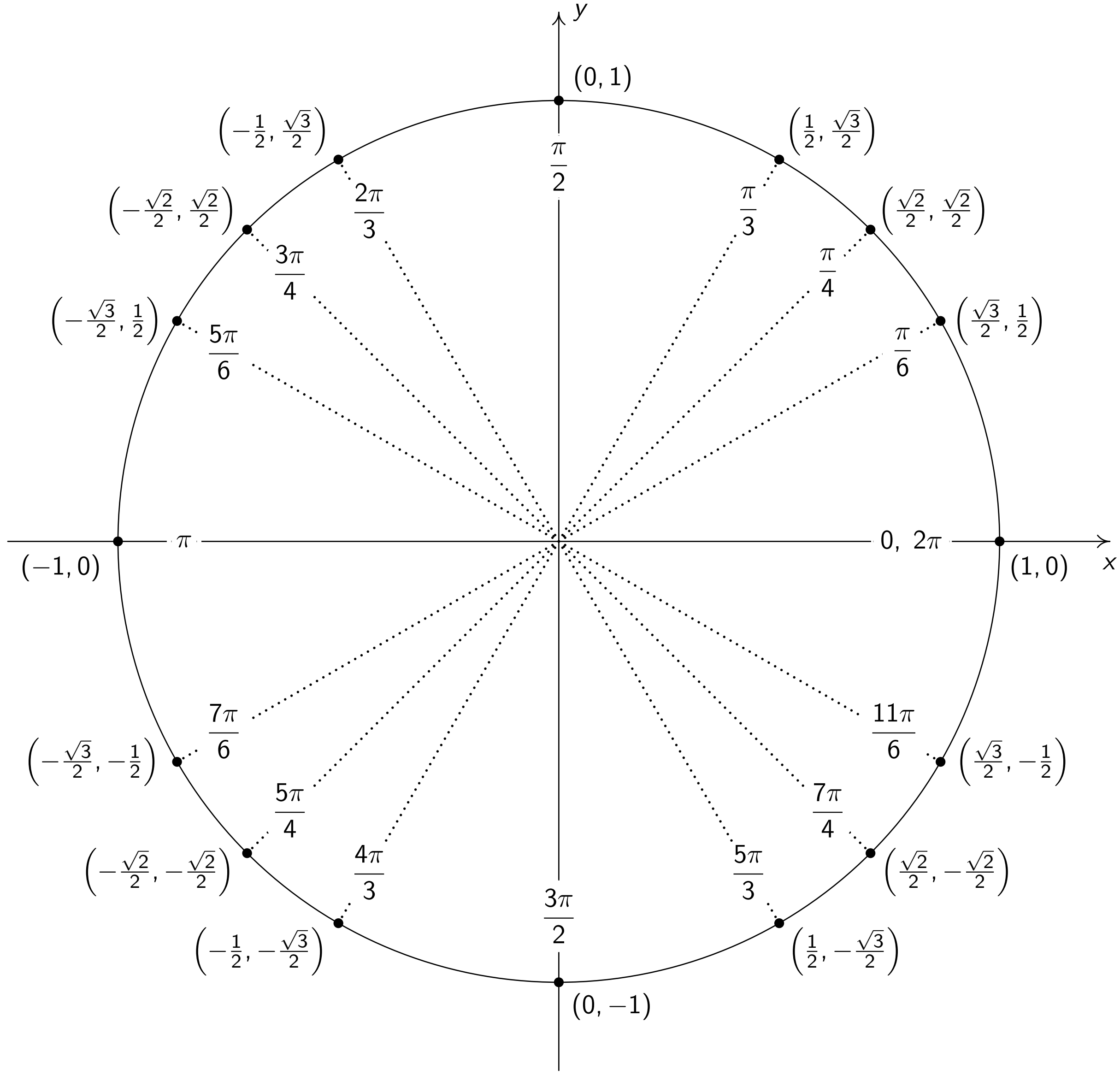
EXAMPLE 3: Suppose α is an acute angle with $\sin(\alpha) = \frac{3}{5}$.

1. Find $\cos(\alpha)$ and use this to plot α in standard position.

2. Find the following

   (a) $\sin(\pi + \alpha)$ (b) $\cos(2\pi - \alpha)$ (c) $\sin(3\pi - \alpha)$

IMPORTANT POINTS ON THE UNIT CIRCLE



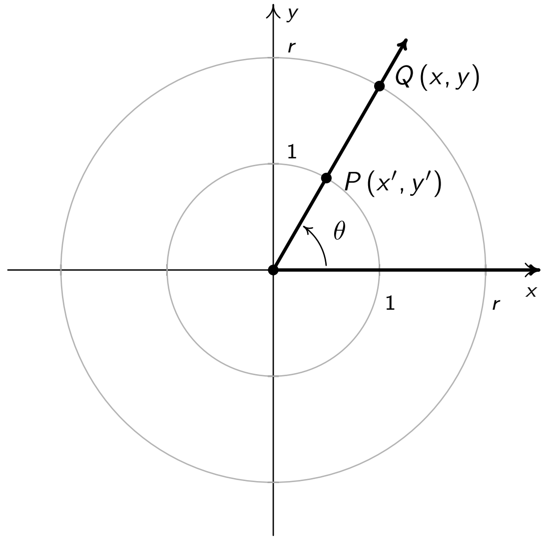
COTERMINAL ANGLES THEOREM:

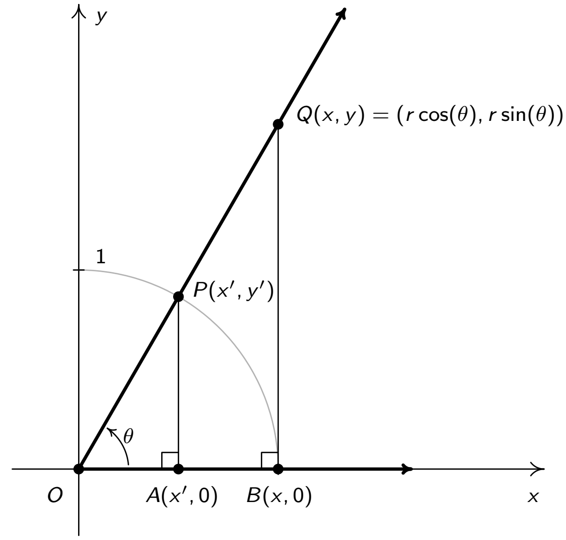
Two angles α and β are coterminal **if and only if** $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$.

EXAMPLE 4: Find all angles that satisfy the following equations. Express your answers in radians.

- $\sin(\theta) = \dfrac{1}{2}$
- $\cos(\alpha) = -\dfrac{1}{2}$
- $\cos(\beta) = 0$
- $\sin(\gamma) = \dfrac{3}{2}$

BEYOND THE UNIT CIRCLE:





THEOREM:

If $Q(x, y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2 + y^2 = r^2$ then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Moreover,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

Note that in the case of the Unit Circle we have $r = \sqrt{x^2+y^2} = 1$, so these generalized formulas specialize to the definitions given at the start of the section.

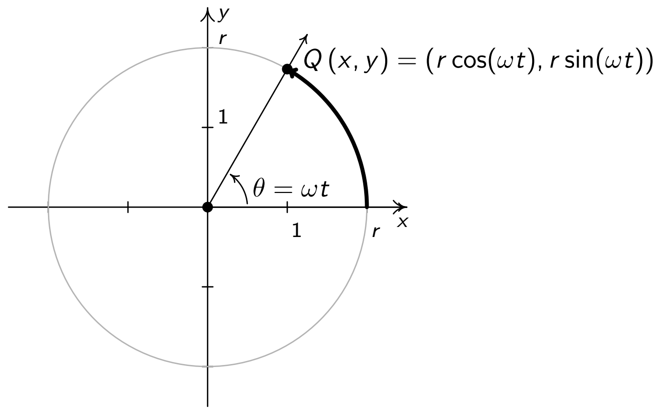
EXAMPLE 5:

1. Find $\sin(\theta)$ if the terminal side of θ, when plotted in standard position, contains the point $Q(1, -2)$.

2. Suppose $\frac{\pi}{2} < \theta < \pi$ with $\sin(\theta) = \frac{5}{13}$. Find $\cos(\theta)$.

EQUATIONS OF CIRCULAR MOTION:

Suppose an object is traveling along a circular path of radius r with constant angular velocity ω. Suppose that at time t, the object has swept out an angle measuring θ radians. Assume that the object is at the point $(r, 0)$ when $t = 0$, the angle θ is in standard position. By definition, $\omega = \frac{\theta}{t}$ which we rewrite as $\theta = \omega t$. Hence, the location of the object $Q(x, y)$ on the circle is found using the equations $x = r\cos(\theta) = r\cos(\omega t)$ and $y = r\sin(\theta) = r\sin(\omega t)$. Hence, at time t, the object is at the point $(r\cos(\omega t), r\sin(\omega t))$, as seen below.



Equations for Circular Motion

EXAMPLE 6:

1. Assuming the Earth is a sphere with radius 3960 miles, verify that the radius of the Earth at 41.628° North latitude is approximately 2960 miles.

2. Find the equations of motion of Lakeland Community College as the earth rotates.

HOMEWORK: Section 10.2: Page 839: 1 - 57 odd, 59-64*